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ANALYSIS OF AN ISOTROPIC FINITE WEDGE UNDER ANTIPLANE DEFORMATION

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Abstract—The antiplane deformation of an isotropic wedge with finite radius is studied in this paper. Depending upon the boundary data prescribed on the circular segment of the wedge, traction or displacement, two problems are analysed. In each problem three different cases of boundary conditions on the radial edges are considered. The radial boundary data are: traction-displacement, displacement-displacement and traction-traction. The solution of governing differential equations is accomplished by means of finite Mellin transforms. The closed form solutions are obtained for displacement and stress fields in the entire domain. The geometric singularities of stress fields are identical to those cited in the literature. However, in displacement-displacement case under certain representation of boundary condition, another type of singularity has been observed. Copyright © 1996 Elsevier Science Ltd

INTRODUCTION

The stress analysis in a wedge with infinite radius has been considered by various investigators. Tranter (1948), by employing Airy stress function and using the Mellin transform, solved the plane elasticity problem of an infinite isotropic wedge. Then, Williams (1952) studied the stress singularities at the wedge apex by using the eigen-function expansion method. Later on, Dempsey and Sinclair (1979), examined the stress singularity at the wedge apex under different loading conditions. In a series of papers, Dempsey (1981) and Ting (1984) and (1985) discussed the paradox which existed in the elementary solution of an elastic wedge. Ting in his work (1985) considered an expansion form of the harmonic eigen-functions and then, by applying the boundary conditions, obtained the coefficients of this expansion.

The analysis of a wedge with finite radius under antiplane deformation is the subject of the present investigation. Two problems related to the type of boundary data on the circular portion of the boundary are studied. The traction free and fixed displacement conditions are imposed on the arc for problems I and II, respectively. The boundary conditions on the radial edges of the wedge in these problems are: displacement-traction, displacement-displacement and traction-traction. The tractions are assumed to act concentrically which allows the solutions to be used as the Green's function for the analysis of a wedge under general distribution of traction. The solution is accomplished by employing the finite Mellin transforms. The full field solution is obtained for displacement and stresses. In all cases, the orders of stress singularity due to wedge geometry are in agreement with the published results in the literature. However, in the displacement-displacement case, depending upon the applied displacement, a new type of stress singularity has been detected on the wedge apex. It is shown, as it was expected, that in the special case of a wedge with infinite radius, the results of the two problems become identical.

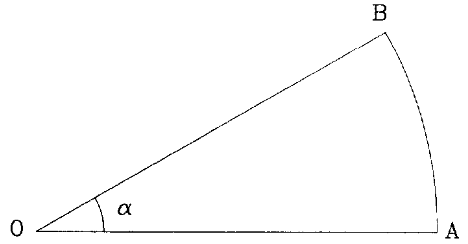

Fig. 1. Schematic view of a finite wedge with radius a and wedge angle α.

FORMULATION AND PROBLEM SOLUTION

A wedge with radius a, apex angle α and infinite length in the direction perpendicular to the plane of the wedge is considered as shown in Fig. 1. The condition of antiplane shear deformation is imposed on the wedge. This implies that the only non-zero displacement component be the out of plane component, W, which is a function of in-plane coordinates r and θ. Therefore, the non-vanishing stress components are $\tau_{rz}(r,\theta)$ and $\tau_{\theta z}(r,\theta)$. The constitutive equations for isotropic materials undergoing antiplane deformation reduce to

$$\tau_{rz} = \mu\frac{\partial W}{\partial r}$$

$$\tau_{\theta z} = \frac{\mu}{r}\frac{\partial W}{\partial \theta} \tag{1}$$

where μ designates the material shear modulus. In the absence of body forces, by making use of (1), the equilibrium equation in terms of displacement appears as

$$\frac{\partial^2 W}{\partial r^2} + \frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2} = 0. \tag{2}$$

One piece of boundary data prevailing in all the cases treated in problem I, i.e., cases Ia, Ib and Ic is the traction free condition on the circular segment of the wedge circumference

$$\tau_{rz}(a,\theta) = 0. \tag{3}$$

In problem II, i.e., cases IIa, IIb and IIc, the wedge is fixed on the circular segment of the boundary. Thus

$$W(a,\theta) = 0. \tag{4}$$

The solution to the Laplace's eqn (2) for a finite wedge may be accomplished by means of the finite Mellin transforms. The finite Mellin transform of first and second kinds are defined, respectively (Sneddon, 1972) as

$$M_1[W(r,\theta), S] = W_1^*(S,\theta) = \int_0^a \left(\frac{a^{2S}}{r^{S+1}} - r^{S-1}\right) W(r,\theta)\,\mathrm{d}r$$

$$M_2[W(r,\theta), S] = W_2^*(S,\theta) = \int_0^a \left(\frac{a^{2S}}{r^{S+1}} + r^{S-1}\right) W(r,\theta)\,\mathrm{d}r \tag{5}$$

where S is a complex transform parameter. The inversions of these transforms are represented by

$$M_j^{-1}[W_j^*(S,\theta),r] = W(r,\theta) = \frac{(-1)^j}{2\pi i}\int_{C-i\infty}^{C+i\infty} r^{-S} W_j^*(S,\theta)\,\mathrm{d}S \quad (j=1,2). \tag{6}$$

The above formula for $j = 1$ differs from that of corresponding Sneddon's equation in a sign. It is an easy task to verify (6). The application of Mellin transform of first kind in conjunction with integration by parts on (2) yields

$$\left(\frac{\partial^2}{\partial\theta^2}+S^2\right)W_1^*(S,\theta)+2Sa^S W(a,\theta) = 0 \tag{7}$$

provided that

$$\lim_{r\to 0}\left[(a^{2S}r^{-S}-r^S)r\frac{\partial W(r,\theta)}{\partial r}+S(a^{2S}r^{-S}+r^S)W(r,\theta)\right] = 0. \tag{8}$$

Similarly, employing the Mellin transform of second kind on (2), leads to

$$\left(\frac{\partial^2}{\partial\theta^2}+S^2\right)W_2^*(S,\theta)+2a^{S+1}\frac{\partial W(a,\theta)}{\partial r} = 0 \tag{9}$$

provided that

$$\lim_{r\to 0}\left[(a^{2S}r^{-S}+r^S)r\frac{\partial W(r,\theta)}{\partial r}+S(a^{2S}r^{-S}-r^S)W(r,\theta)\right] = 0. \tag{10}$$

The conditions expressed by (8) and (10) specify the strip of regularity which is the range of proper values for the real quantity C in the inversion formulas (6). Applying the boundary condition (4) on (7), and the boundary data (3) with the aid of the first of (1) on (9) lead to the following equation for both problems

$$\frac{\mathrm{d}^2 W_j^*}{\mathrm{d}\theta^2}+S^2 W_j^* = 0 \quad (j=1,2). \tag{11}$$

The solution to this equation is readily known to be

$$W_j^*(S,\theta) = A_j(S)\sin(S\theta)+B_j(S)\cos(S\theta) \quad (j=1,2). \tag{12}$$

In the following two problems the boundary data on the radial edges are enforced to compute the unknown coefficients in (12). The values of j are 2 and 1 in problems I and II, respectively.

PROBLEM I

Depending upon the prescribed conditions on the boundary segments OA and OB in Fig. 1, three different cases of traction-displacement, displacement-displacement and traction-traction may be recognized in each problem. These cases for Problem I are analyzed separately in this section.

Case Ia—traction-displacement

Let the wedge be fixed on the boundary OA and subjected to antiplane shear traction on the edge OB. Therefore, the following boundary conditions may be considered

$$\tau_{\theta z}(r,\alpha) = P\delta(r-h) \quad 0 < h < a$$
$$W(r,0) = 0 \tag{13}$$

where δ denotes the Dirac-Delta function. It is worth mentioning that the choice of the first of boundary data (13), leads to the Green's function solution for the problem. The Mellin transform of second kind of the second boundary data (13) gives the first equation for determination of the unknown coefficients in (12). The second equation may be obtained by substituting the first of boundary data (13) into the second of (1) and taking the Mellin transform of the second kind of the resultant equation. The transformed displacement, then, appears as

$$W_2^*(S,\theta) = \frac{P}{\mu}\frac{\sin(S\theta)}{S\cos(S\alpha)}(a^{2S}h^{-S}+h^S). \tag{14}$$

Making use of (6) with $j = 2$, the inversion of (14) results in

$$W(r,\theta) = \frac{P}{2\pi\mu i}\int_{C-i\infty}^{C+i\infty}\frac{\sin(S\theta)}{S\cos(S\alpha)}(a^{2S}h^{-S}+h^S)r^{-S}\,\mathrm{d}s. \tag{15}$$

To obtain the displacement field, contour integration may be used. The integrand is a meromorphic function in S, and two different regions of $r \leqslant h$ and $r \geqslant h$ should be considered. From condition (10) and the requirement that the expression for strain energy ought to be integrable in the vicinity of wedge apex, the strip of regularity for $r \leqslant h$ becomes $|C| < \pi/2\alpha$. We complete the contour of integration by a semi-circular arc to include the negative part of the real axis, $\mathrm{Re}(S) < 0$, where Re stands for the real part of the complex argument. Since the integrand vanishes as $|S| \to \infty$, by utilizing the residue theorem, we obtain the displacement field

$$W(r,\theta) = \frac{P}{\mu}\sum_{k=0}^{\infty}(-1)^k\frac{2}{(2k+1)\pi}\left[1+\left(\frac{h}{a}\right)^{\frac{(2K-1)\pi}{\alpha}}\right]\left(\frac{r}{h}\right)^{\frac{(2k+1)\pi}{2\alpha}}\sin\left(\frac{(2k+1)\pi\theta}{2\alpha}\right) \quad r \leqslant h. \tag{16}$$

By virtue of (1) and (16), stress components read as

$$\tau_{rz}(r,\theta) = \frac{P}{h\alpha}\sum_{k=0}^{\infty}(-1)^k\left[1+\left(\frac{h}{a}\right)^{\frac{(2k+1)\pi}{\alpha}}\right]\left(\frac{r}{h}\right)^{\left(\frac{(2k+1)\pi}{2\alpha}-1\right)}\sin\left(\frac{(2k+1)\pi\theta}{2\alpha}\right) \quad r \leqslant h$$
$$\tau_{\theta z}(r,\theta) = \frac{P}{h\alpha}\sum_{k=0}^{\infty}(-1)^k\left[1+\left(\frac{h}{a}\right)^{\frac{(2k+1)\pi}{\alpha}}\right]\left(\frac{r}{h}\right)^{\left(\frac{(2k+1)\pi}{2\alpha}-1\right)}\cos\left(\frac{(2k+1)\pi\theta}{2\alpha}\right) \quad r \leqslant h. \tag{17}$$

In the region $r \geqslant h$ the integration of each term of (15) should be done individually. The continuity of displacement across the arc $r = h$ requires that the strip of regularity be the same as for $r \leqslant h$. For integrating the first and second terms, we close the path of integration by semi-circular arcs to cover the half-planes $\mathrm{Re}(S) < 0$ and $\mathrm{Re}(S) > 0$, respectively. Consequently, the integrands vanish as $|S| \to \infty$. Making use of the residue theorem leads to

$$W(r,\theta)=\frac{P}{\mu}\sum_{k=0}^{\infty}(-1)^k\frac{2}{(2k+1)\pi}\left[1+\left(\frac{r}{a}\right)^{\frac{(2k+1)\pi}{\alpha}}\right]\left(\frac{h}{r}\right)^{\frac{(2k+1)\pi}{2\alpha}}\sin\left(\frac{(2k+1)\pi\theta}{2\alpha}\right)\quad r\geqslant h. \tag{18}$$

Plugging (18) into the constitutive relationships (1) the stress components are determined

$$\tau_{rz}(r,\theta)=\frac{P}{h\alpha}\sum_{k=0}^{\infty}(-1)^k\left[1-\left(\frac{r}{a}\right)^{\frac{(2k+1)\pi}{\alpha}}\right]\left(\frac{h}{r}\right)^{\left(\frac{(2k+1)\pi}{2\alpha}+1\right)}\sin\left(\frac{(2k+1)\pi\theta}{2\alpha}\right)\quad r\geqslant h$$

$$\tau_{\theta z}(r,\theta)=\frac{P}{h\alpha}\sum_{k=0}^{\infty}(-1)^k\left[1+\left(\frac{r}{a}\right)^{\frac{(2k+1)\pi}{\alpha}}\right]\left(\frac{h}{r}\right)^{\left(\frac{(2k+1)\pi}{2\alpha}+1\right)}\cos\left(\frac{(2k+1)\pi\theta}{2\alpha}\right)\quad r\geqslant h. \tag{19}$$

Equations (16)–(19) show that the series solutions for displacement and stress component τ_{rz} are divergent at the point of application of traction. Moreover the value of τ_{rz} is discontinuous on the arc $r=h$. From (17), it is observed that the stress fields are bounded in a wedge with $0<\alpha<\pi/2$, whereas in a wedge with $\pi/2<\alpha<2\pi$, we have

$$(\tau_{rz},\tau_{\theta z})=O(r^{-\lambda})\quad \text{as}\quad r\to 0$$

and the strength of geometric singularity

$$\lambda=1-\frac{\pi}{2\alpha}$$

which is in accord with the investigation of Ma and Hour (1989). Furthermore, for $\alpha=\pi$ and $\alpha=2\pi$ the strength of singularities becomes 1/2 and 3/4, respectively. These are exactly the same stress singularities obtained by Ting (1986) in wedges with the foregoing apex angles but undergoing in-plane loading conditions. We may also mention that due to the symmetry, the analysis of a circular isotropic shaft with a radial crack under anti-plane shear traction on the crack flanks reduces to this case where the apex angle $\alpha=\pi$.

In the particular case of a wedge with infinite radius, the displacement and stress fields can be obtained by letting $a\to\infty$ in (16)–(19). For the sake of brevity, only the displacement component is indicated

$$W(r,\theta)=\frac{P}{\mu}\sum_{k=0}^{\infty}(-1)^k\frac{2}{(2k+1)\pi}\left(\frac{r}{h}\right)^{\frac{(2k+1)\pi}{2\alpha}}\sin\left(\frac{(2k+1)\pi\theta}{2\alpha}\right)\quad r\leqslant h$$

$$W(r,\theta)=\frac{P}{\mu}\sum_{k=0}^{\infty}(-1)^k\frac{2}{(2k+1)\pi}\left(\frac{h}{r}\right)^{\frac{(2k+1)\pi}{2\alpha}}\sin\left(\frac{(2k+1)\pi\theta}{2\alpha}\right)\quad r\geqslant h \tag{20}$$

Case Ib—displacement-displacement

We consider a wedge fixed on the boundary OA and subjected to antiplane deformation on the edge OB, Fig. 1. Thus the boundary conditions for (12) may have the following form

$$W(r,\theta)=0$$
$$W(r,\alpha)=r^n\quad n>0 \tag{21}$$

where n is a real constant. It is noteworthy to indicate that in general any displacement boundary data on the edge OB may be represented by its Taylor series expansion at $r=0$.

The second boundary condition (21) is the general form of a term of such series. From the boundary data (21) and the second of (5), the coefficients in (12) are determined. The transformed displacement reduces to

$$W_2^*(S,\theta) = \frac{2na^{n+S}}{n^2-S^2}\frac{\sin(S\theta)}{\sin(S\alpha)}. \tag{22}$$

Using the inversion formula (6), yields

$$W(r,\theta) = \frac{2na^n}{2\pi i}\int_{C-i\infty}^{C+i\infty}\left(\frac{a}{r}\right)^S \frac{\sin(S\theta)}{(n^2-S^2)\sin(S\alpha)}\,\mathrm{d}s. \tag{23}$$

From the limit condition (10), the strip of regularity is $\mathrm{Max}(-\pi/\alpha, -n) < C < \mathrm{Min}(\pi/\alpha, n)$ and the path of integration is chosen to contain the second and third quadrants of complex S-plane. In a wedge with angle $\alpha \neq m\pi/n$ where m is a positive integer, all the singularities of the integrand in (23) are simple poles and the displacement may be derived by utilizing the residue theorem

$$W(r,\theta) = 2na^n\left[\left(\frac{r}{a}\right)^n\frac{\sin(n\theta)}{2n\sin(n\alpha)} + \sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{\alpha}\left(\frac{r}{a}\right)^{\frac{k\pi}{\alpha}}\frac{\sin\left(\frac{k\pi\theta}{\alpha}\right)}{\left(n^2-\frac{k^2\pi^2}{\alpha^2}\right)}\right]. \tag{24}$$

By virtue of (1), stress fields are obtained as

$$\tau_{rz}(r,\theta) = 2n\mu a^{n-1}\left[\left(\frac{r}{a}\right)^{n-1}\frac{\sin(n\theta)}{2\sin(n\alpha)} + \sum_{k=1}^{\infty}(-1)^{k+1}\frac{k\pi}{\alpha^2}\left(\frac{r}{a}\right)^{\frac{k\pi}{\alpha}-1}\frac{\sin\left(\frac{k\pi\theta}{\alpha}\right)}{n^2-\left(\frac{k\pi}{\alpha}\right)^2}\right]$$

$$\tau_{\theta z}(r,\theta) = 2n\mu a^{n-1}\left[\left(\frac{r}{a}\right)^{n-1}\frac{\cos(n\theta)}{2\sin(n\alpha)} + \sum_{k=1}^{\infty}(-1)^{k+1}\frac{k\pi}{\alpha^2}\left(\frac{r}{a}\right)^{\frac{k\pi}{\alpha}-1}\frac{\cos\left(\frac{k\pi\theta}{\alpha}\right)}{n^2-\left(\frac{k\pi}{\alpha}\right)^2}\right]. \tag{25}$$

From (25) on the arc segment $r = a$, we have

$$\tau_{rz}(a,\theta) = 2n\mu a^{n-1}\left[\frac{\sin(n\theta)}{2\sin(n\alpha)} + \sum_{k=1}^{\infty}(-1)^{k+1}\frac{k\pi}{\alpha^2}\frac{\sin\left(\frac{k\pi\theta}{\alpha}\right)}{\left(n^2-\frac{k^2\pi^2}{\alpha^2}\right)}\right]. \tag{26}$$

The satisfaction of traction free condition on $r = a$ is not apparent from (26). In order to verify this condition, it suffices to expand $\sin(n\theta)$ in the first term of (26) by a Fourier sine series with period 2π (Spiegel, 1968)

$$\sin(n\theta) = \frac{2}{\pi}\sin(n\alpha)\sum_{k=1}^{\infty}(-1)^k \frac{k}{k^2-\left(\frac{n\alpha}{\pi}\right)^2}\sin\frac{k\pi\theta}{\alpha}. \tag{27}$$

Substitution of above equality into (26) reproduces the boundary condition (3).

The first terms of stress solutions (25) are singular as $r \to 0$ where $0 < n < 1$. This type of singularity is induced by the severity of applied displacement on the boundary $\theta = \alpha$ near the wedge apex and may be regarded as load singularity. The first terms of the series parts of solutions in (25) are singular at the apex, where $\alpha > \pi$ and the strength of geometric singularity may be calculated from the following relationship

$$\lambda = 1 - \frac{\pi}{\alpha}.$$

Clearly the displacement and stress fields (24) and (25) become unbounded in the whole region as $\alpha \to m\pi/n$. The integrand in (23) has a double pole at $S = -m\pi/\alpha$ and the other singularities are simple poles. Nonetheless, the previous path of integration is still applicable. Carrying out the contour integration, the displacement is determined

$$W(r,\theta) = 2na^n\left[\frac{(-1)^{(n\alpha/\pi)+1}}{2n\alpha}\left(\frac{r}{a}\right)^n\left[\left(\ln\left(\frac{a}{r}\right)+\frac{1}{2n}\right)\sin(n\theta)-\theta\cos(n\theta)\right] + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{\alpha}\left(\frac{r}{a}\right)^{\frac{k\pi}{\alpha}}\frac{\sin\left(\frac{k\pi\theta}{\alpha}\right)}{n^2-\left(\frac{k\pi}{\alpha}\right)^2}\right]_{k\neq m}. \tag{28}$$

The stress components from (1) result in

$$\tau_{rz}(r,\theta) = \frac{2na^{n-1}}{\alpha}\mu\left[\frac{(-1)^{n\alpha/\pi}}{2}\left(\frac{r}{a}\right)^{n-1}\left[\left(\ln\left(\frac{r}{a}\right)+\frac{1}{2n}\right)\sin(n\theta)+\theta\cos(n\theta)\right] + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}k\pi}{\alpha}\left(\frac{r}{a}\right)^{\frac{k\pi}{\alpha}-1}\frac{\sin\left(\frac{k\pi\theta}{\alpha}\right)}{n^2-\left(\frac{k\pi}{\alpha}\right)^2}\right]_{k\neq m}$$

$$\tau_{\theta z}(r,\theta) = \frac{2na^{n-1}}{\alpha}\mu\left[\frac{(-1)^{n\alpha/\pi}}{2}\left(\frac{r}{a}\right)^{n-1}\left[\left(\ln\left(\frac{r}{a}\right)+\frac{1}{2n}\right)\cos(n\theta)-\theta\sin(n\theta)\right] + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}k\pi}{\alpha}\left(\frac{r}{a}\right)^{\frac{k\pi}{\alpha}-1}\frac{\cos\left(\frac{k\pi\theta}{\alpha}\right)}{n^2-\left(\frac{k\pi}{\alpha}\right)^2}\right]_{k\neq m}. \tag{29}$$

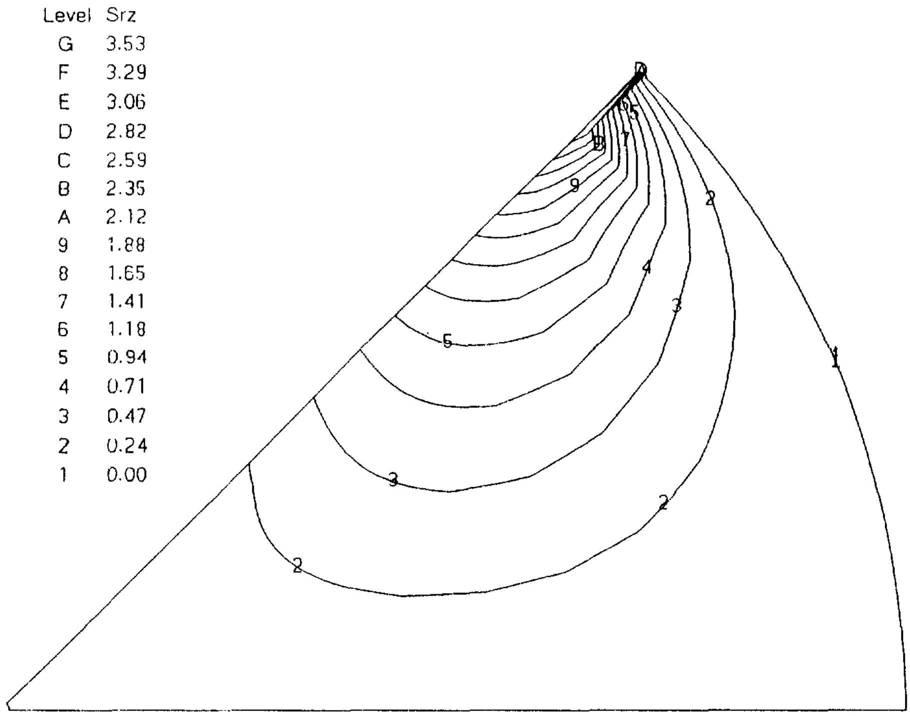

Fig. 2. Contours of nondimensionalized shear stress τ_{rz}/μ.

For $n < 1$, the first terms of (29) have a singularity of the order $(r^{(n-1)} \ln r)$ as $r \to 0$ which is caused by the applied boundary data. Dempsey and Sinclair (1979) reported this type of singularity for a composite wedge under in-plane loading. Furthermore, when $n = 1$, stress components (29) exhibit logarithmic singularity. Sinclair (1980) predicted this type of behaviour for the temperature field at the apex of a composite wedge in the steady-state heat conduction problem. For $m \neq 1$, the series parts of (29) become singular as $r \to 0$ and the strength of geometric singularity is identical with that of (25). When $m = 1$ the geometric singularity vanishes. In order to verify the satisfaction of boundary data (3), utilizing the first of (29), the nondimensionalized stress contours τ_{rz}/μ are plotted for a wedge with $\alpha = \pi/4$ under displacement boundary data $\mathbf{W}(r, \pi/4) = r^4$, Fig. 2. As we may observe, $\tau_{rz}(a, \theta)$ vanishes. For any other values of α and n which satisfy $\alpha = m\pi/n$, by plotting the stress contours τ_{rz}/μ, the foregoing conclusion may be reached.

Case Ic—traction-traction

Let the wedge be subjected to antiplane shear tractions on boundary segments OA and OB, Fig. 1. The boundary conditions are considered as

$$\tau_{\theta z}(r, 0) = P\delta(r - h_2)$$
$$\tau_{\theta z}(r, \alpha) = P\delta(r - h_1). \tag{30}$$

Without loss of generality, we assume that $h_1 \leqslant h_2$. Taking the Mellin transform of second kind of the second of eqns (1), we deduce that

$$\frac{\mathrm{d}W_2(S, \theta)}{\mathrm{d}\theta} = \frac{1}{\mu} \int_0^a \left(\frac{a^{2S}}{r^S} + r^S \right) \tau_{\theta z}(r, \theta)\, \mathrm{d}r. \tag{31}$$

Applying the boundary conditions (30) to (31), we arrive at

$$\frac{\mathrm{d}W_2^*(S,0)}{\mathrm{d}\theta}=\frac{P}{\mu}(a^{2S}h_2^{-S}+h_2^S)$$

$$\frac{\mathrm{d}W_2^*(S,\alpha)}{\mathrm{d}\theta}=\frac{P}{\mu}(a^{2S}h_1^{-S}+h_1^S) \tag{32}$$

From (32) the coefficients in (12) with $j = 2$, may be calculated and the Mellin transform of displacement is

$$W_2^*(S,\theta)=\frac{P}{\mu S}\Big\{(a^{2S}h_2^{-S}+h_2^S)\sin(S\theta)$$
$$+[(a^{2S}h_2^{-S}+h_2^S)\cos(S\alpha)-(a^{2S}h_1^{-S}+h_1^S)]\frac{\cos(S\theta)}{\sin(S\alpha)}\Big\}. \tag{33}$$

Eliminating the analytic terms in (33) (analytic terms have no contribution in the contour integration) and using inversion formula (6), we have

$$W(r,\theta)=\frac{P}{2\pi i\mu}\int_{C-i\infty}^{C-i\infty}r^{-S}[(a^{2S}h_2^{-S}+h_2^S)\cos(S\alpha)-(a^{2S}h_1^{-S}+h_1^S)]\frac{\cos(S\theta)}{S\sin(S\alpha)}\mathrm{d}S. \tag{34}$$

In the region $r \leqslant h_1$ from the limit condition (10), the strip of regularity becomes $-\pi/\alpha < C < 0$. The appropriate contour of integration is a semicircular arc which engulfs the second and third quadrants of complex S-plane. The result of contour integration yields

$$W(r,\theta)=-\frac{P}{\mu}\sum_{k=1}^{\infty}\frac{1}{k\pi}\Big\{(-1)^{k+1}\Big[1+\Big(\frac{h_1}{a}\Big)^{\frac{2k\pi}{\alpha}}\Big]\Big(\frac{r}{h_1}\Big)^{\frac{k\pi}{\alpha}}$$
$$+\Big[1+\Big(\frac{h_2}{a}\Big)^{\frac{2k\pi}{\alpha}}\Big]\Big(\frac{r}{h_2}\Big)^{\frac{k\pi}{\alpha}}\Big\}\cos\Big(\frac{k\pi\theta}{\alpha}\Big)$$
$$r \leqslant h_1. \tag{35}$$

The stress fields are determined from (1)

$$\tau_{rz}(r,\theta)=\frac{P}{h_1\alpha}\sum_{k=1}^{\infty}\Big\{(-1)^k\Big(\frac{r}{h_1}\Big)^{\left(\frac{k\pi}{\alpha}-1\right)}\Big[1+\Big(\frac{h_1}{a}\Big)^{\frac{2k\pi}{\alpha}}\Big]$$
$$-\Big(\frac{h_1}{h_2}\Big)\Big(\frac{r}{h_2}\Big)^{\left(\frac{k\pi}{\alpha}-1\right)}\Big[1+\Big(\frac{h_2}{a}\Big)^{\frac{2k\pi}{\alpha}}\Big]\Big\}\cos\Big(\frac{k\pi\theta}{\alpha}\Big)\quad r\leqslant h_1$$

$$\tau_{\theta z}(r,\theta)=\frac{P}{h_1\alpha}\sum_{k=1}^{\infty}\Big\{(-1)^{k+1}\Big(\frac{r}{h_1}\Big)^{\left(\frac{k\pi}{\alpha}-1\right)}\Big[1+\Big(\frac{h_1}{a}\Big)^{\frac{2k\pi}{\alpha}}\Big]$$
$$+\Big(\frac{h_1}{h_2}\Big)\Big(\frac{r}{h_2}\Big)^{\left(\frac{k\pi}{\alpha}-1\right)}\Big[1+\Big(\frac{h_2}{a}\Big)^{\frac{2k\pi}{\alpha}}\Big]\Big\}\sin\Big(\frac{k\pi\theta}{\alpha}\Big)\quad r\leqslant h_1 \tag{36}$$

In order to carry out the integrations in (34) for the regions $h_1 \leqslant r \leqslant h_2$ and $r \leqslant h_2$, depending upon the terms under consideration, the contour of integration should contain either the first and fourth or the second and third quadrants of the complex S-plane. The choice of contour is subjected to the requirement that the integrand should approach zero as $|S| \to \infty$. The continuity of displacement across the arcs $r = h_1$ and $r = h_2$, implies that the strip of regularity is $0 < C < \pi/\alpha$ for the former contour whereas it is $-\pi/\alpha < C < 0$ for the latter one. The results of contour integration may be written as

$$W(r,\theta) = -\frac{P}{\mu}\sum_{k=1}^{\infty}\frac{1}{k\pi}\left[(-1)^{k+1}\left(\frac{r}{a}\right)^{\frac{k\pi}{\alpha}}\left(\frac{h_1}{a}\right)^{\frac{k\pi}{\alpha}}+\left(\frac{r}{h_2}\right)^{\frac{k\pi}{\alpha}}\left(1+\left(\frac{h_2}{a}\right)^{\frac{2k\pi}{\alpha}}\right)\right.$$
$$\left.+(-1)^{k+1}\left(\frac{h_1}{r}\right)^{\frac{k\pi}{\alpha}}\right]\cos\left(\frac{k\pi\theta}{\alpha}\right) \quad h_1 \leqslant r \leqslant h_2$$

$$W(r,\theta) = -\frac{P}{\mu}\sum_{k=1}^{\infty}\frac{1}{k\pi}\left[\left(\frac{r}{a}\right)^{\frac{k\pi}{\alpha}}\left[(-1)^{k+1}\left(\frac{h_1}{a}\right)^{\frac{k\pi}{\alpha}}+\left(\frac{h_2}{a}\right)^{\frac{k\pi}{\alpha}}\right]\right.$$
$$\left.+(-1)^{k+1}\left(\frac{h_1}{r}\right)^{\frac{k\pi}{\alpha}}+\left(\frac{h_2}{r}\right)^{\frac{k\pi}{\alpha}}\right]\cos\left(\frac{k\pi\theta}{\alpha}\right) \quad r \geqslant h_2. \tag{37}$$

Substitution of (37) into the constitutive eqns (1) gives the stress fields

$$\tau_{rz}(r,\theta) = \frac{P}{h_1\alpha}\sum_{k=1}^{\infty}\left[(-1)^{k}\left(\frac{r}{a}\right)^{\frac{k\pi}{\alpha}-1}\left(\frac{h_1}{a}\right)^{\frac{k\pi}{\alpha}+1}-\left(\frac{h_1}{h_2}\right)\left(\frac{r}{h_2}\right)^{\frac{k\pi}{\alpha}-1}\left[1+\left(\frac{h_2}{a}\right)^{\frac{2k\pi}{\alpha}}\right]\right.$$
$$\left.+(-1)^{k+1}\left(\frac{h_1}{r}\right)^{\frac{k\pi}{\alpha}+1}\right]\cos\left(\frac{k\pi\theta}{\alpha}\right) \quad h_1 \leqslant r \leqslant h_2$$

$$\tau_{\theta z}(r,\theta) = \frac{P}{h_1\alpha}\sum_{k=1}^{\infty}\left[(-1)^{k+1}\left(\frac{r}{a}\right)^{\frac{k\pi}{\alpha}-1}\left(\frac{h_1}{a}\right)^{\frac{k\pi}{\alpha}+1}+\left(\frac{h_1}{h_2}\right)\left(\frac{r}{h_2}\right)^{\frac{k\pi}{\alpha}-1}\left[1+\left(\frac{h_2}{a}\right)^{\frac{2k\pi}{\alpha}}\right]\right.$$
$$\left.+(-1)^{k+1}\left(\frac{h_1}{r}\right)^{\frac{k\pi}{\alpha}+1}\right]\sin\left(\frac{k\pi\theta}{\alpha}\right) \quad h_1 \leqslant r \leqslant h_2$$

$$\tau_{rz}(r,\theta) = \frac{P}{h_1\alpha}\sum_{k=1}^{\infty}\left[(-1)^{k}\left(\frac{r}{a}\right)^{\frac{k\pi}{\alpha}-1}\left(\frac{h_1}{a}\right)^{\frac{k\pi}{\alpha}+1}-\left(\frac{h_1}{h_2}\right)\left(\frac{r}{a}\right)^{\frac{k\pi}{\alpha}-1}\left(\frac{h_2}{a}\right)^{\frac{k\pi+1}{\alpha}}+\left(\frac{h_1}{r}\right)^{\frac{k\pi}{\alpha}+1}\right.$$
$$\left.+\left(\frac{h_1}{h_2}\right)\left(\frac{h_2}{r}\right)^{\frac{k\pi}{\alpha}+1}\right]\cos\left(\frac{k\pi\theta}{\alpha}\right) \quad r \geqslant h_2$$

$$\tau_{\theta z}(r,\theta) = \frac{P}{h_1\alpha}\sum_{k=1}^{\infty}\left[(-1)^{k+1}\left(\frac{r}{a}\right)^{\frac{k\pi}{\alpha}-1}\left(\frac{h_1}{a}\right)^{\frac{k\pi}{\alpha}+1}+\left(\frac{h_1}{h_2}\right)\left(\frac{r}{a}\right)^{\frac{k\pi}{\alpha}-1}\left(\frac{h_2}{a}\right)^{\frac{k\pi}{\alpha}+1}+(-1)^{k+1}\left(\frac{h_1}{r}\right)^{\frac{k\pi}{\alpha}+1}\right.$$
$$\left.+\left(\frac{h_1}{h_2}\right)\left(\frac{h_2}{r}\right)^{\frac{k\pi}{\alpha}+1}\right]\sin\left(\frac{k\pi\theta}{\alpha}\right) \quad r \geqslant h_2 \tag{38}$$

In the particular case of $h_1 = h_2$, the line $\theta = \alpha/2$ is the line of symmetry. Consequently, $W(r, \alpha/2) = 0$, and we may observe that the solutions (35)–(38) are in agreement with those of a wedge under traction-displacement boundary condition, case Ia, with apex angle half of the wedge considered here.

Analogous to the traction-displacement case, the series solutions for displacement and stress field, τ_{rz}, are divergent at the points of application of tractions. Furthermore, τ_{rz} is discontinuous on the arcs $r = h_1$ and $r = h_2$. From (36) the stress components τ_{rz} and $\tau_{\theta z}$

are bounded in a wedge with apex angle $0 < \alpha \leqslant \pi$. The strength of geometric singularity in a wedge with angle $\pi < \alpha \leqslant 2\pi$ is

$$\lambda = 1 - \frac{\pi}{\alpha}$$

which is identical with the geometric singularity of case Ib. For $\alpha = 2\pi$, the wedge resembles a circular shaft with a radial crack under antiplane shear stresses on the crack faces, and the stress fields exhibit the familiar square root singularity.

In a wedge with infinite radius the displacement and stress fields may be obtained by taking the limit of (35)–(38) as $a \to \infty$. Here, the displacement component is merely mentioned.

$$W(r,\theta) = -\frac{P}{\mu}\sum_{k=1}^{\infty}\frac{1}{k\pi}\left(\frac{r}{h_2}\right)^{\frac{k\pi}{\alpha}}\left[1+(-1)^{k+1}\left(\frac{h_2}{h_1}\right)^{\frac{k\pi}{\alpha}}\right]\cos\left(\frac{k\pi\theta}{\alpha}\right) \quad r \leqslant h_1$$

$$W(r,\theta) = -\frac{P}{\mu}\sum_{k=1}^{\infty}\frac{1}{k\pi}\left(\frac{r}{h_2}\right)^{\frac{k\pi}{\alpha}}\left[1+(-1)^{k+1}\left(\frac{h_1 h_2}{r^2}\right)^{\frac{k\pi}{\alpha}}\right]\cos\left(\frac{k\pi\theta}{\alpha}\right) \quad h_1 \leqslant r \leqslant h_2$$

$$W(r,\theta) = -\frac{P}{\mu}\sum_{k=1}^{\infty}\frac{1}{k\pi}\left(\frac{h_2}{r}\right)^{\frac{k\pi}{\alpha}}\left[1+(-1)^{k+1}\left(\frac{h_1}{h_2}\right)^{\frac{k\pi}{\alpha}}\right]\cos\left(\frac{k\pi\theta}{\alpha}\right) \quad r \geqslant h_2. \tag{39}$$

PROBLEM II

Basically the analysis of problem II parallels that of Problem I. Therefore, the analysis has been made brief in this problem. In the sequel, three foregoing cases of boundary data on the radial edges of the wedge are taken into account.

Case IIa—traction-displacement

The boundary data on the radial edges are denoted by (13). The application of these conditions with the aid of the second of (1) on (12) with $j = 1$, results in

$$W_1^*(S,\theta) = \frac{P}{\mu}\frac{\sin(S\theta)}{S\cos(S\alpha)}(a^{2S}h^{-S} - h^S) \tag{40}$$

From the inversion formula (6) with $j = 1$ and (40), we have

$$W(r,\theta) = -\frac{P}{2\pi\mu i}\int_{C-i\infty}^{C+i\infty}\frac{\sin(S\theta)}{S\cos(S\alpha)}(a^{2S}h^{-S} - h^S)r^{-S}\,\mathrm{d}s. \tag{41}$$

The condition (8) dictates that the strip of regularity be analogous to that of case Ia and the contour integration follows the same line of calculations

$$W(r,\theta) = \frac{P}{\mu}\sum_{k=0}^{\infty}(-1)^k\frac{2}{(2k+1)\pi}\left[1-\left(\frac{h}{a}\right)^{\frac{(2k+1)\pi}{\alpha}}\right]\left(\frac{r}{h}\right)^{\frac{(2k+1)\pi}{2\alpha}}\sin\left(\frac{(2k+1)\pi\theta}{2\alpha}\right) \quad r \geqslant h$$

$$W(r,\theta) = \frac{P}{\mu}\sum_{k=0}^{\infty}(-1)^k\frac{2}{(2k+1)\pi}\left[1-\left(\frac{r}{a}\right)^{\frac{(2k+1)\pi}{\alpha}}\right]\left(\frac{h}{r}\right)^{\frac{(2k+1)\pi}{2\alpha}}\sin\left(\frac{(2k+1)\pi\theta}{2\alpha}\right) \quad r \leqslant h. \tag{42}$$

The substitution of (42) into constitutive relationships, (1), gives the stress fields

$$\tau_{rz}(r,\theta) = \frac{P}{h\alpha}\sum_{k=0}^{\infty}(-1)^{k+1}\left[1-\left(\frac{h}{a}\right)^{\frac{(2k+1)\pi}{\alpha}}\right]\left(\frac{r}{h}\right)^{\frac{(2k+1)\pi}{2\alpha}-1}\sin\left(\frac{(2k+1)\pi\theta}{2\alpha}\right) \quad r \leqslant h$$

$$\tau_{\theta z}(r,\theta) = \frac{P}{h\alpha}\sum_{k=0}^{\infty}(-1)^{k+1}\left[1-\left(\frac{h}{a}\right)^{\frac{(2k+1)\pi}{\alpha}}\right]\left(\frac{r}{h}\right)^{\frac{(2k+1)\pi}{2\alpha}-1}\cos\left(\frac{(2k+1)\pi\theta}{2\alpha}\right) \quad r \leqslant h$$

$$\tau_{rz}(r,\theta) = \frac{P}{h\alpha}\sum_{k=0}^{\infty}(-1)^{k}\left[1+\left(\frac{r}{a}\right)^{\frac{(2k-1)\pi}{\alpha}}\right]\left(\frac{h}{a}\right)^{\frac{(2k+1)\pi}{2\alpha}+1}\sin\left(\frac{(2k+1)\pi\theta}{2\alpha}\right) \quad r \geqslant h$$

$$\tau_{\theta z}(r,\theta) = \frac{P}{h\alpha}\sum_{k=0}^{\infty}(-1)^{k+1}\left[1+\left(\frac{r}{a}\right)^{\frac{(2k+1)\pi}{\alpha}}\right]\left(\frac{h}{r}\right)^{\frac{(2k+1)\pi}{2\alpha}+1}\cos\left(\frac{(2k+1)\pi\theta}{2\alpha}\right) \quad r \geqslant h. \tag{43}$$

Turning our attention to a wedge with infinite radius, it is obvious that since at infinity the displacement and stress fields tend to zero, this case should convert to case Ia. To verify the statement we may easily let $a \to \infty$ in (42) and reproduce (20).

Case IIb—displacement-displacement

In a manner similar to case Ib, we take the Mellin transform of the first kind of boundary data (21) and use the results to compute the coefficients in (12) with $j = 1$. The Mellin transform of displacement is then

$$W_1^*(S,\theta) = \frac{2Sa^{n+2}}{n^2-S^2}\frac{\sin(S\theta)}{\sin(S\alpha)}. \tag{44}$$

Applying the inversion formula, (6), to this equation gives

$$W(r,\theta) = \frac{-a^n}{\pi i}\int_{C-i\infty}^{C+i\infty}\left(\frac{a}{r}\right)^S\frac{S\sin(S\theta)}{(n^2-S^2)\sin(S\alpha)}\,\mathrm{d}s. \tag{45}$$

The above integration in a wedge with angle $\alpha \neq m\pi/n$ (m has integral values) is carried out by employing the contour of integration indicated in case Ib. The displacement field becomes

$$W(r,\theta) = 2a^n\left[\left(\frac{r}{a}\right)^n\frac{\sin(n\theta)}{2\sin(n\alpha)}+\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k\pi}{\alpha^2}\left(\frac{r}{a}\right)^{\frac{k\pi}{\alpha}}\frac{\sin\left(\frac{k\pi\theta}{\alpha}\right)}{\left(n^2-\frac{k^2\pi^2}{\alpha^2}\right)}\right]. \tag{46}$$

By inserting (27) into (46), it can be shown that the boundary data, (4), is satisfied. The stress fields from (1) yield

$$\tau_{rz}(r,\theta) = 2\mu a^{n-1}\left[n\left(\frac{r}{a}\right)^{n-1}\frac{\sin(n\theta)}{2\sin(n\alpha)}+\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k^2\pi^2}{\alpha^3}\left(\frac{r}{a}\right)^{\frac{k\pi-1}{\alpha}}\frac{\sin\left(\frac{k\pi\theta}{\alpha}\right)}{n^2-\left(\frac{k\pi}{\alpha}\right)^2}\right]$$

$$\tau_{\theta z}(r,\theta) = 2\mu a^{n-1}\left[n\left(\frac{r}{a}\right)^{n-1}\frac{\cos(n\theta)}{2\sin(n\alpha)}+\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k^2\pi^2}{\alpha^3}\left(\frac{r}{a}\right)^{\frac{k\pi-1}{\alpha}}\frac{\cos\left(\frac{k\pi\theta}{\alpha}\right)}{n^2-\left(\frac{k\pi}{\alpha}\right)^2}\right] \tag{47}$$

When $\alpha = m\pi/n$, the integrand in (45) has a double pole at $S = -m\pi/\alpha$. Carrying out the contour integration, we get

$$W(r,\theta) = 2a^n \left[\frac{(-1)^{(n\alpha/\pi)+1}}{2\alpha} \left(\frac{r}{a}\right)^n \left[\left[\ln\left(\frac{a}{r}\right) - \frac{1}{2n} \right] \sin(n\theta) - \theta\cos(n\theta) \right] + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} k\pi}{\alpha^2} \left(\frac{r}{a}\right)^{\frac{k\pi}{\alpha}} \frac{\sin\left(\frac{k\pi\theta}{\alpha}\right)}{n^2 - \left(\frac{k\pi}{\alpha}\right)^2} \right]_{k\neq m} . \tag{48}$$

We should note that setting $r = a$ in the first of (29) and in (48), results in

$$W(a,\theta) = (a/\mu)\tau_{rz}(a,\theta). \tag{49}$$

Consequently, we may use Fig. (2) to argue that the boundary condition, (4), is satisfied. The constitutive eqns (1) and displacement solution (48) give the stress fields

$$\tau_{rz}(r,\theta) = \frac{2a^{n-1}}{\alpha}\mu \left[\frac{(-1)^{n\alpha/\pi} n}{2} \left(\frac{r}{a}\right)^{n-1} \left[\left[\ln\left(\frac{r}{a}\right) + \frac{3}{2n} \right] \sin(n\theta) + \theta\cos(n\theta) \right] + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} k^2\pi^2}{\alpha 2} \left(\frac{r}{a}\right)^{\frac{k\pi}{\alpha}-1} \frac{\sin\left(\frac{k\pi\theta}{\alpha}\right)}{n^2 - \left(\frac{k\pi}{\alpha}\right)^2} \right]_{k\neq m}$$

$$\tau_{\theta z}(r,\theta) = \frac{2a^{n-1}}{\alpha}\mu \left[\frac{(-1)^{n\alpha/\pi} n}{2} \left(\frac{r}{a}\right)^{n-1} \left[\left[\ln\left(\frac{r}{a}\right) + \frac{3}{2n} \right] \cos(n\theta) - \theta\sin(n\theta) \right] + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} k^2\pi^2}{\alpha 2} \left(\frac{r}{a}\right)^{\frac{k\pi}{\alpha}-1} \frac{\cos\left(\frac{k\pi\theta}{\alpha}\right)}{n^2 - \left(\frac{k\pi}{\alpha}\right)^2} \right]_{k\neq m} \tag{50}$$

Case IIc—traction-traction

Taking the Mellin transforms of first kind of the second of (1), we have

$$\frac{\mathrm{d}W_1^*(S,\theta)}{\mathrm{d}\theta} = \frac{1}{\mu} \int_0^a \left(\frac{a^{2S}}{r^S} - r^S \right) \tau_{\theta z}(r,\theta)\, \mathrm{d}r. \tag{51}$$

Similar to case Ic, applying the boundary conditions, (30), with the aid of (51) on (12) where $j = 1$, yields

$$W_1^*(S,\theta) = -\frac{P}{\mu S}\left\{(a^{2S}h_2^{-S}-h_2^S)\sin(S\theta) + [(a^{2S}h_2^{-S}-h_2^S)\cos(S\alpha)-(a^{2S}h_1^{-S}-h_1^S)]\frac{\cos(S\theta)}{\sin(S\alpha)}\right\}. \quad (52)$$

From (6) and (52) after eliminating the analytic terms, we have

$$W(r,\theta) = -\frac{P}{2\pi i\mu}\int_{C-i\infty}^{C+i\infty} r^{-S}[(a^{2S}h_2^{-S}-h_2^S)\cos(S\alpha)-(a^{2S}h_1^{-S}-h_1^S)]\frac{\cos(S\theta)}{S\sin(S\alpha)}\,\mathrm{d}S. \quad (53)$$

The condition (8) makes it necessary that the strip of regularity be the same as that in case Ic. Moreover, the same contour of integration should be used. Performing the contour integration, the displacement in the whole region yields

$$W(r,\theta) = -\frac{P}{\mu}\sum_{k=1}^{\infty}\frac{1}{k\pi}\left[(-1)^{k+1}\left[1-\left(\frac{h_1}{a}\right)^{\frac{2k\pi}{\alpha}}\right]\left(\frac{r}{h_1}\right)^{\frac{k\pi}{\alpha}} + \left[1-\left(\frac{h_2}{a}\right)^{\frac{2k\pi}{\alpha}}\right]\left(\frac{r}{h_2}\right)^{\frac{k\pi}{\alpha}}\right]\cos\left(\frac{k\pi\theta}{\alpha}\right) \quad r \leqslant h_1$$

$$W(r,\theta) = -\frac{P}{\mu}\sum_{k=1}^{\infty}\frac{1}{k\pi}\left[(-1)^k\left(\frac{r}{a}\right)^{\frac{k\pi}{\alpha}}\left(\frac{h_1}{a}\right)^{\frac{k\pi}{\alpha}} + \left(\frac{r}{h_2}\right)^{\frac{k\pi}{\alpha}}\left(1-\left(\frac{h_2}{a}\right)^{\frac{2k\pi}{\alpha}}\right) + (-1)^{k+1}\left(\frac{h_1}{r}\right)^{\frac{k\pi}{\alpha}}\right]\cos\left(\frac{k\pi\theta}{\alpha}\right) \quad h_1 \leqslant r \leqslant h_2$$

$$W(r,\theta) = -\frac{P}{\mu}\sum_{k=1}^{\infty}\frac{1}{k\pi}\left[\left(\frac{r}{a}\right)^{\frac{k\pi}{\alpha}}\left[(-1)^k\left(\frac{h_1}{a}\right)^{\frac{k\pi}{\alpha}}-\left(\frac{h_2}{a}\right)^{\frac{k\pi}{\alpha}}\right] + \left[(-1)^{k+1}\left(\frac{h_1}{r}\right)^{\frac{k\pi}{\alpha}}+\left(\frac{h_2}{r}\right)^{\frac{k\pi}{\alpha}}\right]\right]\cos\left(\frac{k\pi\theta}{\alpha}\right) \quad r \geqslant h_2. \quad (54)$$

Substitution of (54) into (1) leads to the stress fields

$$\tau_{rz}(r,\theta) = \frac{P}{h_1\alpha}\sum_{k=1}^{\infty}\left[(-1)^k\left(\frac{r}{h_1}\right)^{\left(\frac{k\pi}{\alpha}-1\right)}\left[1-\left(\frac{h_1}{a}\right)^{\frac{2k\pi}{\alpha}}\right] + \left(\frac{h_1}{h_2}\right)\left(\frac{r}{h_2}\right)^{\left(\frac{k\pi}{\alpha}-1\right)}\left[1-\left(\frac{h_2}{a}\right)^{\frac{2k\pi}{\alpha}}\right]\right]\cos\left(\frac{k\pi\theta}{\alpha}\right) \quad r \leqslant h_1$$

$$\tau_{\theta z}(r,\theta) = \frac{P}{h_1\alpha}\sum_{k=1}^{\infty}\left[(-1)^{k+1}\left(\frac{r}{h_1}\right)^{\left(\frac{k\pi}{\alpha}-1\right)}\left[1-\left(\frac{h_1}{a}\right)^{\frac{2k\pi}{\alpha}}\right] + \left(\frac{h_1}{h_2}\right)\left(\frac{r}{h_2}\right)^{\left(\frac{k\pi}{\alpha}-1\right)}\left[1-\left(\frac{h_2}{a}\right)^{\frac{2k\pi}{\alpha}}\right]\right]\sin\left(\frac{k\pi\theta}{\alpha}\right) \quad r \leqslant h_1$$

$$\tau_{rz}(r,\theta) = \frac{P}{h_1\alpha}\sum_{k=1}^{\infty}\left[(-1)^{k+1}\left(\frac{r}{a}\right)^{\frac{k\pi}{\alpha}-1}\left(\frac{h_1}{a}\right)^{\frac{k\pi}{\alpha}+1}+\left(\frac{h_1}{h_2}\right)\left(\frac{r}{h_2}\right)^{\frac{k\pi}{\alpha}-1}\left[1-\left(\frac{h_2}{a}\right)^{\frac{2k\pi}{\alpha}}\right]\right.$$
$$\left.+(-1)^{k+1}\left(\frac{h_1}{r}\right)^{\frac{k\pi}{\alpha}+1}\right]\cos\left(\frac{k\pi\theta}{\alpha}\right) \quad h_1 \leqslant r \leqslant h_2$$

$$\tau_{\theta z}(r,\theta) = \frac{P}{h_1\alpha}\sum_{k=1}^{\infty}\left[(-1)^{k}\left(\frac{r}{a}\right)^{\frac{k\pi}{\alpha}-1}\left(\frac{h_1}{a}\right)^{\frac{k\pi}{\alpha}+1}+\left(\frac{h_1}{h_2}\right)\left(\frac{r}{h_2}\right)^{\frac{k\pi}{\alpha}-1}\left[1-\left(\frac{h_2}{a}\right)^{\frac{2k\pi}{\alpha}}\right]\right.$$
$$\left.+(-1)^{k+1}\left(\frac{h_1}{r}\right)^{\frac{k\pi}{\alpha}+1}\right]\sin\left(\frac{k\pi\theta}{\alpha}\right) \quad h_1 \leqslant r \leqslant h_2$$

$$\tau_{rz}(r,\theta) = \frac{P}{h_1\alpha}\sum_{k=1}^{\infty}\left[(-1)^{k+1}\left(\frac{r}{a}\right)^{\frac{k\pi}{\alpha}-1}\left(\frac{h_1}{a}\right)^{\frac{k\pi}{\alpha}+1}+\left(\frac{h_1}{h_2}\right)\left(\frac{r}{a}\right)^{\frac{k\pi}{\alpha}-1}\left(\frac{h_2}{a}\right)^{\frac{k\pi}{\alpha}+1}+(-1)^{k+1}\left(\frac{h_1}{r}\right)^{\frac{k\pi}{\alpha}+1}\right.$$
$$\left.+\left(\frac{h_1}{h_2}\right)\left(\frac{h_2}{r}\right)^{\frac{k\pi}{\alpha}+1}\right]\cos\left(\frac{k\pi\theta}{\alpha}\right) \quad r \geqslant h_2$$

$$\tau_{\theta z}(r,\theta) = \frac{P}{h_1\alpha}\sum_{k=1}^{\infty}\left[(-1)^{k}\left(\frac{r}{a}\right)^{\frac{k\pi}{\alpha}-1}\left(\frac{h_1}{a}\right)^{\frac{k\pi}{\alpha}+1}-\left(\frac{r}{a}\right)^{\frac{k\pi}{\alpha}-1}\left(\frac{h_2}{a}\right)^{\frac{k\pi}{\alpha}}\left(\frac{h_1}{a}\right)+\left(\frac{h_2}{r}\right)^{\frac{k\pi}{\alpha}-1}\left(\frac{h_1}{h_2}\right)\right.$$
$$\left.+(-1)^{k+1}\left(\frac{h_1}{r}\right)^{\frac{k\pi}{\alpha}+1}\right]\sin\left(\frac{k\pi\theta}{\alpha}\right) \quad r \geqslant h_2 \qquad (55)$$

In a wedge with infinite radius, by the reasoning stated in case IIa, the displacement field should be the same as that of case Ic. Taking the limit as $a \to \infty$ in (54), we obtain (39).

The discussions regarding the behavior of displacement and stress fields in the neighborhood of the point of application of traction and the singularities of stress fields in different cases of problem II are the same as those in the corresponding cases of problem I.

CONCLUSIONS

The stress analysis of a finite wedge under antiplane shear deformation has been investigated in this paper. The finite Mellin transform is employed to solve the governing differential equation. The boundary data on the circular arc in problem I is traction free and in problem II is the zero displacement component. All possible boundary conditions on the radial edges are taken into account. These cases are traction-displacement, displacement-displacement and traction-traction. Exact closed form solution for each above-mentioned case is obtained for displacement and stress fields. The effects of apex angle upon strength of singularity of stress fields in each case have been discussed. For the special case of a wedge with infinite radius, the results of the two problems are identical.

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REFERENCES

Dempsey, J. P. (1981). The wedge subjected to tractions: a paradox resolved. *J. Elasticity* **11**, 1–10.

Dempsey, J. P. and Sinclair, G. B. (1979). On the stress singularities in the plane elasticity of the composite wedge. *J. Elasticity* **9**, 373–391.

Ma, C. C. and Hour, B. L. (1990). Antiplane problems in composite anisotropic materials with an inclined crack terminating at a bimaterial interface. *Int. J. Solids Structures* **26**, 1387–1400.

Sinclair, G. B. (1980). On the singular eigen-functions for plane harmonic problems in composite regions. *J. Appl. Mech.* **47,** 87–92.
Sneddon, I. N. (1972). *The Use of Integral Transforms*, McGraw-Hill Book Co., New York.
Spiegel, M. R. (1968). *Mathematical Handbook*, McGraw-Hill Book Co., New York.
Ting, T. C. T. (1984). The wedge subjected to tractions: a paradox re-examined. *J. Elasticity* **14,** 235–247.
Ting, T. C. T. (1985). Elastic wedge subjected to antiplane shear traction—a paradox explained. *Q. J. Mech. Appl. Math.* **38,** 245–255.
Ting, T. C. T. (1986). Explicit solution and invariance of the singularities at an interface crack in anisotropic composites. *Int. J. Solids Structures* **22,** 965–983.
Tranter, C. J. (1948). The use of the Mellin transform in finding the stress distribution in an infinite wedge. *Q. J. Mech. Appl. Math.* **1,** 125–130.
Williams, M. L. (1952). Stress singularities resulting from various boundary conditions in angular corners of plates in extension. *J. Appl. Mech.* **19,** 526–528.